ON TENSORIAL CHARACTERISTICS OF FINITE DEFORMATIONS OF A CONTINUOUS MEDIUM

(O TENZORNYKH KHARAKTERISTIKAKH KONECHNYKH DEFORMATSII SPLOSHNOI SREDY)

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The description of finite deformations of a continuous medium is related to the use of tensors in the initial-state and the deformed-state spaces with identical arrangements of indices (see [1]). In these two spaces the identical Lagrangian coordinates ξ^1, ξ^2, ξ^3 of the particles of the medium can be introduced with the covariant and contravariant base vectors $\hat{\partial}_i$ and $\hat{\partial}^i$ ($i = 1, 2, 3$) in the deformed-state space or with the base vectors $\overset{\circ}{\partial}_i$ and $\overset{\circ}{\partial}{}^i$ in the initial-state space. The metrics of the two considered spaces are different. If $d\overset{\circ}{s}$ and ds denote the elementary segments in the initial and the deformed states, respectively, we have

$$d\overset{\circ}{s}{}^2 = \overset{\circ}{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta, \qquad ds^2 = \hat{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta \qquad (\overset{\circ}{g}_{\alpha\beta} = (\overset{\circ}{\partial}_\alpha, \overset{\circ}{\partial}_\beta), \ \hat{g}_{\alpha\beta} = (\hat{\partial}_\alpha, \hat{\partial}_\beta)) \quad (0.1)$$

We shall consider tensors as invariant objects related to the particles of the medium subjected to a deformation process. Consider an arbitrary tensor of the second rank in the deformed space

$$H = \hat{H}_{\alpha\beta} \hat{\partial}^\alpha \hat{\partial}^\beta = \hat{H}^\alpha{}_\beta \hat{\partial}_\alpha \hat{\partial}^\beta = \hat{H}_\alpha{}^\beta \hat{\partial}^\alpha \hat{\partial}_\beta = \hat{H}^{\alpha\beta} \hat{\partial}_\alpha \hat{\partial}_\beta \qquad (0.2)$$

In the initial space the tensors $\overset{\circ}{H}_1$, $\overset{\circ}{H}_2$, $\overset{\circ}{H}_3$ and $\overset{\circ}{H}_4$ can be established whose covariant, mixed, and contravariant components are equal to the respective components of the tensor H

$$\overset{\circ}{H}_1 = \hat{H}_{\alpha\beta} \overset{\circ}{\partial}{}^\alpha \overset{\circ}{\partial}{}^\beta, \qquad \overset{\circ}{H}_2 = \hat{H}^\alpha{}_\beta \overset{\circ}{\partial}_\alpha \overset{\circ}{\partial}{}^\beta, \qquad \overset{\circ}{H}_3 = \hat{H}_\alpha{}^\beta \overset{\circ}{\partial}{}^\alpha \overset{\circ}{\partial}_\beta, \qquad \overset{\circ}{H}_4 = \hat{H}^{\alpha\beta} \overset{\circ}{\partial}_\alpha \overset{\circ}{\partial}_\beta. \quad (0.3)$$

Since the rising and lowering of indices employs different tensors, $g_{\alpha\beta}$ and $\overset{\circ}{g}_{\alpha\beta}$, in the deformed and the initial spaces, the components of tensors $\overset{\circ}{H}_i$, different from these shown above, are not equal to the corresponding components of the tensor H. Similarly, to an arbitrary tensor $\overset{\circ}{H}$ in the initial space

$$\mathring{H} = \mathring{H}_{\alpha\beta}\mathring{\partial}^{\alpha}\mathring{\partial}^{\beta} = \mathring{H}^{\alpha}{}_{\beta}\mathring{\partial}_{\alpha}\mathring{\partial}^{\beta} = \mathring{H}_{\alpha}{}^{\beta}\mathring{\partial}^{\alpha}\mathring{\partial}_{\beta} = \mathring{H}^{\alpha\beta}\mathring{\partial}_{\alpha}\mathring{\partial}_{\beta} \tag{0.4}$$

correspond four tensors in the deformed space

$$H_1 = \mathring{H}_{\alpha\beta}\hat{\partial}^{\alpha}\hat{\partial}^{\beta}, \qquad H_2 = \mathring{H}^{\alpha}{}_{\beta}\hat{\partial}_{\alpha}\hat{\partial}^{\beta}, \qquad H_3 = \mathring{H}_{\alpha}{}^{\beta}\hat{\partial}^{\alpha}\hat{\partial}_{\beta}, \qquad H_4 = \mathring{H}^{\alpha\beta}\hat{\partial}_{\alpha}\hat{\partial}_{\beta} \tag{0.5}$$

which have analogous properties. Furthermore, similar considerations can be repeated with respect to the tensors H_i and $\mathring{H}_i$ (i = 1, 2, 3, 4), and thus new groups of tensors can be established, etc. As a result, for the original tensors H and $\mathring{H}$ we obtain an infinite sequence of tensors in the initial and the deformed spaces. In the following, we shall investigate the laws governing the construction of the tensors in the sequences established for the metric tensors, the tensors of finite deformation, and for the derivatives of the tensors of the second order. We shall show the applications of new tensorial characteristics in the derivation of the equations of state of the medium and in the definitions of the rates of stress.

1. Let us consider the metric tensors $\mathring{G}$ and G of the initial and the deformed states. According to Expressions (0.1) we have

$$\begin{gathered}\mathring{G} = \mathring{g}_{\alpha\beta}\mathring{\partial}^{\alpha}\mathring{\partial}^{\beta} = \delta_{\beta}{}^{\alpha}\mathring{\partial}_{\alpha}\mathring{\partial}^{\beta} = \mathring{g}^{\alpha\beta}\mathring{\partial}_{\alpha}\mathring{\partial}_{\beta}, \quad G = \hat{g}_{\alpha\beta}\hat{\partial}^{\alpha}\hat{\partial}^{\beta} = \delta_{\beta}{}^{\alpha}\hat{\partial}_{\alpha}\hat{\partial}^{\beta} = \hat{g}^{\alpha\beta}\hat{\partial}_{\alpha}\hat{\partial}_{\beta} \\ \mathring{g}^{\alpha\beta} = (\mathring{\partial}^{\alpha}, \mathring{\partial}^{\beta}), \quad \hat{g}^{\alpha\beta} = (\hat{\partial}^{\alpha}, \hat{\partial}^{\beta}), \quad \delta_{\beta}{}^{\alpha} = (\mathring{\partial}^{\alpha}, \mathring{\partial}_{\beta}) = (\hat{\partial}^{\alpha}, \hat{\partial}_{\beta}) = \begin{cases}1 & (\alpha = \beta)\\ 0 & (\alpha \neq \beta)\end{cases}\end{gathered} \tag{1.1}$$

The following tensors correspond to the tensor G in the initial space:

$$\begin{gathered}\mathring{G} = \delta_{\beta}{}^{\alpha}\mathring{\partial}_{\alpha}\mathring{\partial}^{\beta}, \quad \mathring{A} = \hat{g}_{\alpha\beta}\mathring{\partial}^{\alpha}\mathring{\partial}^{\beta} = \mathring{g}^{\alpha\sigma}\hat{g}_{\sigma\beta}\mathring{\partial}_{\alpha}\mathring{\partial}^{\beta} = \mathring{g}^{\alpha\sigma}\hat{g}_{\sigma k}\mathring{g}^{k\beta}\mathring{\partial}_{\alpha}\mathring{\partial}_{\beta} \\ \mathring{B} = \hat{g}^{\alpha\beta}\mathring{\partial}_{\alpha}\mathring{\partial}_{\beta} = \hat{g}^{\alpha\sigma}\mathring{g}_{\sigma\beta}\mathring{\partial}_{\alpha}\mathring{\partial}^{\beta} = \mathring{g}_{\alpha k}\hat{g}^{k\sigma}\mathring{g}_{\sigma\beta}\mathring{\partial}^{\alpha}\mathring{\partial}^{\beta}\end{gathered} \tag{1.2}$$

and the following tensors correspond to the tensor $\mathring{G}$ in the deformed space:

$$\begin{gathered}G = \delta_{\beta}{}^{\alpha}\hat{\partial}_{\alpha}\hat{\partial}^{\beta}, \quad A = \mathring{g}_{\alpha\beta}\hat{\partial}^{\alpha}\hat{\partial}^{\beta} = \hat{g}^{\alpha\sigma}\mathring{g}_{\sigma\beta}\hat{\partial}_{\alpha}\hat{\partial}^{\beta} = \hat{g}^{\alpha\sigma}\mathring{g}_{\sigma k}\hat{g}^{k\beta}\hat{\partial}_{\alpha}\hat{\partial}_{\beta} \\ B = \mathring{g}^{\alpha\beta}\hat{\partial}_{\alpha}\hat{\partial}_{\beta} = \mathring{g}^{\alpha\sigma}\hat{g}_{\sigma\beta}\hat{\partial}_{\alpha}\hat{\partial}^{\beta} = \hat{g}_{\alpha k}\mathring{g}^{k\sigma}\hat{g}_{\sigma\beta}\hat{\partial}^{\alpha}\hat{\partial}^{\beta}\end{gathered} \tag{1.3}$$

As we see, the tensors having mixed components equal to the mixed components of G and $\mathring{G}$ are also metric tensors; the tensors $\mathring{A}(A)$ and $\mathring{B}(B)$ having covariant and contravariant components equal to the respective components of $G(G)$ are new tensors. The new tensors are obviously symmetric; they are, respectively, inverse: $\mathring{B} = \mathring{A}^{-1}$, $B = A^{-1}$ and the mixed components of the tensors $\mathring{A} = \| \mathring{a}_j{}^i \|$ and $B = \| \hat{b}_j{}^i \|$, and $A = \| \hat{a}_j{}^i \|$ $\mathring{B} = \| \mathring{b}_j{}^i \|$ are equal

$$\mathring{a}_j{}^i = \hat{b}_j{}^i = \mathring{g}^{i\sigma}\hat{g}_{\sigma j}, \qquad \mathring{b}_j{}^i = \hat{a}_j{}^i = \hat{g}^{i\sigma}\mathring{g}_{\sigma j} \tag{1.4}$$

The following tensors correspond, in turn, to the tensors A, A^{-1} and $\mathring{A}$, $\mathring{A}^{-1}$ in the initial and the deformed states:

$$\mathring{B} = \mathring{b}_\beta{}^\alpha,\ \mathring{\partial}_\alpha \mathring{\partial}^\beta, \qquad \mathring{G} = \mathring{g}_{\alpha\beta}\mathring{\partial}^\alpha \mathring{\partial}^\beta, \qquad \mathring{B}_2 = \mathring{b}_k{}^\alpha \mathring{b}_e{}^k \mathring{g}^{e\beta}\mathring{\partial}_\alpha \mathring{\partial}_\beta \tag{1.5}$$

$$\mathring{A} = \mathring{a}_\beta{}^\alpha \mathring{\partial}_\alpha \mathring{\partial}^\beta, \qquad \mathring{A}_2 = \mathring{g}_{\alpha e}\mathring{a}_\sigma{}^e \mathring{a}_\beta{}^\sigma \mathring{\partial}^\alpha \mathring{\partial}^\beta, \qquad \mathring{G} = \mathring{g}^{\alpha\beta}\mathring{\partial}_\alpha \mathring{\partial}_\beta \tag{1.6}$$

$$B = \hat{b}_\beta{}^\alpha \hat{\partial}_\alpha \hat{\partial}^\beta, \qquad G = \hat{g}_{\alpha\beta}\hat{\partial}^\alpha \hat{\partial}^\beta, \qquad B_2 = \hat{b}_k{}^\alpha \hat{b}_e{}^k \hat{g}^{e\beta}\hat{\partial}_\alpha \hat{\partial}_\beta \tag{1.7}$$

$$A = \hat{a}_\beta{}^\alpha \hat{\partial}_\alpha \hat{\partial}^\beta, \qquad A_2 = \hat{g}_{\alpha e}\hat{a}_\sigma{}^e \hat{a}_\beta{}^\sigma \hat{\partial}^\alpha \hat{\partial}^\beta, \qquad G = \hat{g}^{\alpha\beta}\hat{\partial}_\alpha \hat{\partial}_\beta \tag{1.8}$$

from which only the tensors $\mathring{A}_2$, $\mathring{B}_2$, A_2 and B_2 are new. For them we have the relations

$$\mathring{A}_2 = \mathring{A}^2, \qquad \mathring{B}_2 = \mathring{B}^2 = \mathring{A}^{-2}, \qquad A_2 = A^2, \qquad B_2 = B^2 = A^{-2} \tag{1.9}$$

i.e. the new tensors can be expressed as the powers of the tensor $\mathring{A}$ in the initial state and the tensor A in the deformed state.

It is easy to notice the law obeyed by the tensors which are obtained according to the indicated order of establishing the correspondence. Thus, to the tensor A, for instance, correspond three tensors $\mathring{A}^{-1}$, $\mathring{G}$, $\mathring{A}^{-2}$ in the initial space. The first tensor $\mathring{A}^{-1}$, whose mixed components are the same as those of the tensor A, has the exponent equal in absolute value to that of A, but with the opposite sign. The second tensor $\mathring{G}$, whose covariant components are the same as those of the tensor A, is obtained as the product of the first tensor and the tensor $\mathring{A}$. The third tensor $\mathring{A}^{-2}$, whose contravariant components are the same as those of the tensor A, is obtained as the product of the first tensor and the tensor A^{-1}.

It is easy to notice that this rule holds also for the tensors (1.2) corresponding to the tensor G, and for the tensors (1.6) corresponding to the tensor A^{-1}. Analogous properties exist in the case of tensors (1.3), (1.7), and (1.8) of the deformed space, corresponding to the tensors $\mathring{G}$, $\mathring{A}$ and $\mathring{A}^{-1}$, respectively.

This discussion shows that the process of formation of new tensors consists of separate steps. Therefore, the rule outlined above will be proved if its validity is shown for the n + 1th step, assuming that it is valid for the nth step.

According to the preceding discussion, at the nth step we have the tensors $\mathring{A}^{-n+1}$, $\mathring{A}^{-n+2}$, $\mathring{A}^{-n}$; $\mathring{A}^{n-1}$, $\mathring{A}^{n}$, $\mathring{A}^{n-2}$ in the initial space, and the tensors A^{-n+1}, A^{-n+2}, A^{-n}; A^{n-1}, A^{n}, A^{n-2} in the deformed space, with the tensors $\mathring{A}^{-n}$, $\mathring{A}^{n}$, A^{-n} and A^{n} being new (the other tensors at this step have already been discussed). Let us consider now the tensors corresponding to these new tensors.

To the tensor

$$A^{-n} = \hat{B}_j{}^i \hat{\partial}_i \hat{\partial}^j = \hat{g}_{i\alpha} \hat{B}_j{}^{\alpha} \hat{\partial}^i \hat{\partial}^j = \hat{B}_{\alpha}{}^i \hat{g}^{\alpha j} \hat{\partial}_i \hat{\partial}_j, \qquad \hat{B}_j{}^i = \hat{b}_{\sigma_1}{}^i \hat{b}_{\sigma_2}{}^{\sigma_1} \ldots \hat{b}_j^{\sigma_{n-1}} \quad (1.10)$$

correspond three tensors in the initial space: the tensor with the same mixed components is $\mathring{A}^{n}$, according to (1.4) and (1.10); the tensors with the same covariant or contravariant components as A^{-n} have the mixed components equal to $\mathring{a}_{\alpha}{}^{i} \hat{B}_j{}^{\alpha}$ or $\hat{B}_{\alpha}{}^{i} \mathring{b}_j{}^{\alpha}$, i.e. they are obtained as the products of the tensor $\mathring{A}^{n}$ and the tensor $\mathring{A}$ or the tensor $\mathring{A}^{-1}$. To the tensor A^{-n} correspond thus the tensors $\mathring{A}^{n}$, $\mathring{A}^{n+1}$, $\mathring{A}^{n-1}$.

Similar discussion shows that to the tensor A^{n} correspond the three tensors $\mathring{A}^{-n}$, $\mathring{A}^{-n+1}$, $\mathring{A}^{-n-1}$ in the initial space and, furthermore, to the tensors $\mathring{A}^{-n}$ and $\mathring{A}^{n}$ correspond the tensors A^{n}, A^{n+1}, A^{n-1} and A^{-n}, A^{-n+1}, A^{-n-1} in the deformed space. Consequently, the rule has been proved.

We arrive thus at the conclusion that all the tensors obtained from the metric tensors according to the scheme indicated above are the terms of the sequences

$$\mathring{A}^{k}, \qquad A^{k} \quad (1.11)$$

where k is a positive or negative integer, and $|k|$ denotes the number of the step which yields a given tensor of the sequence for the first time.

It is easy to see that the tensors (1.11) are symmetric and, in addition, the tensors $\mathring{A}^{k}$ and A^{-k} have identical mixed components, which implies that their principal values and invariants are also identical. The tensors $\mathring{A}^{k}$ and A^{-k+1} have identical covariant components, and the tensors $\mathring{A}^{-k}$ and A^{k-1} have identical contravariant components.

We denote by $\mathring{a}_i$ and $\hat{a}_i$ $(i = 1, 2, 3)$ the principal values of the tensors $\mathring{A}$ and A, respectively. The above discussion leads to the conclusion that the tensors (1.11) reduce simultaneously to the principal axes and their principal values are $\mathring{a}_i^{\,k}$ and $\hat{a}_i^{\,k}$ $(i = 1, 2, 3)$, respectively.

The quantities $\mathring{a}_i$ and $\hat{a}_i$ are not independent. As mentioned, the principal values of the tensors A and $\mathring{A}^{-1}$ are identical and, therefore

$$\hat{a}_i = \mathring{a}_i^{-1} \qquad (i = 1, 2, 3) \quad (1.12)$$

We shall now consider the tensors which characterize finite deformations of a continuous medium. As a measure of strain, the following difference is usually assumed:

$$ds^2 - d\mathring{s}^2 = 2\varepsilon_{\alpha\beta} d\xi^\alpha d\xi^\beta, \qquad \varepsilon_{\alpha\beta} = \frac{1}{2}(\hat{g}_{\alpha\beta} - \mathring{g}_{\alpha\beta}) \tag{1.13}$$

and two tensors of finite strain are considered: the tensor $\mathring{\mathscr{E}}$ in the initial space and the tensor $\mathscr{E}$ in the deformed space. Their covariant components are equal [2] :

$$\mathring{\mathscr{E}} = \varepsilon_{\alpha\beta}\mathring{\partial}^\alpha\mathring{\partial}^\beta, \qquad \mathscr{E} = \varepsilon_{\alpha\beta}\hat{\partial}^\alpha\hat{\partial}^\beta \tag{1.14}$$

These tensors are, as can easily be checked, linear functions of the tensors $\mathring{A}$ and A, respectively:

$$\mathring{\mathscr{E}} = \frac{1}{2}(\mathring{A} - \mathring{G}), \qquad \mathscr{E} = \frac{1}{2}(G - A) \tag{1.15}$$

The reversed relations have the form

$$\mathring{A} = \mathring{G} + 2\mathring{\mathscr{E}}, \qquad A = G - 2\mathscr{E} \tag{1.16}$$

Using the method explained above, it is possible to establish the sequences of tensors corresponding to the tensor $\mathring{\mathscr{E}}$ and $\mathscr{E}$ in the initial and the deformed states. With the relations (1.15) and (1.16) and the results of the preceding discussion, it is possible to show that these sequences have the form

$$(\mathring{G} + 2\mathring{\mathscr{E}})^k\mathring{\mathscr{E}}, \qquad (G - 2\mathscr{E})^k\mathscr{E} \qquad (k = \ldots -2, -1, 0, 1, 2, \ldots) \tag{1.17}$$

At each step we obtain two new tensors in the initial space and two tensors in the deformed space. Thus, at the nth step, the new tensors are those corresponding to the following values of k : $k = (-1)^n n$, $(-1)^n n - 1$.

We see that all the tensors of the sequences (1.17) are isotropic tensor functions of the tensors $\mathring{\mathscr{E}}$ and $\mathscr{E}$, respectively, and consequently they are characteristics of finite deformations of a continuous medium.

We shall note some properties of the tensors (1.17), which are easily deduced by the investigation of the relations between tensors in different spaces. All the tensors (1.17) are symmetric. The tensors

$$(\mathring{G} + 2\mathring{\mathscr{E}})^m\mathring{\mathscr{E}}, \qquad (G - 2\mathscr{E})^{-m}\mathscr{E} \qquad (m = \ldots . -2, -1, 0, 1, 2, \ldots) \tag{1.18}$$

have identical covariant components, the tensors

$$(\mathring{G} + 2\mathring{\mathscr{E}})^{m-1}\mathring{\mathscr{E}}, \qquad (G - 2\mathscr{E})^{-m-1}\mathscr{E} \qquad (m = \ldots -2, -1, 0, 1, 2, \ldots) \tag{1.19}$$

have identical contravariant components, and the tensors

$$(\overset{\circ}{G} + 2\overset{\circ}{\mathscr{E}})^{-m}\overset{\circ}{\mathscr{E}}, \qquad (G - 2\mathscr{E})^{m-1}\mathscr{E} \qquad (m = \ldots -2, -1, 0, 1, 2, \ldots) \quad (1.20)$$

have identical mixed components and, consequently, identical principal values and invariants.

In particular, for $m = 0$, the tensors $\overset{\circ}{\mathscr{E}}$ and $\mathscr{E}$ have identical covariant components $\epsilon_{\alpha\beta}$ determined by the relations (1.13). The tensors $\overset{\circ}{\Theta} = (\overset{\circ}{G} + 2\overset{\circ}{\mathscr{E}})^{-1}\overset{\circ}{\mathscr{E}}$ and $\Theta = (G - 2\mathscr{E})^{-1}\mathscr{E}$ have identical contravariant components

$$\theta^{\alpha\beta} = \frac{1}{2}\,(\overset{\circ}{g}{}^{\alpha\beta} - \hat{g}^{\alpha\beta}) \qquad (1.21)$$

and the tensors $\mathscr{E}$ and Θ have identical mixed components

$$\overset{\circ}{\varepsilon}{}^{\alpha}{}_{\beta} = \hat{\theta}^{\alpha}{}_{\beta} = \frac{1}{2}\,(\overset{\circ}{g}{}^{\alpha\sigma}\hat{g}_{\sigma\beta} - \delta^{\alpha}{}_{\beta}), \qquad \overset{\circ}{\varepsilon}{}_{\alpha}{}^{\beta} = \hat{\theta}_{\alpha}{}^{\beta} = \frac{1}{2}\,(\hat{g}_{\alpha\sigma}\overset{\circ}{g}{}^{\sigma\beta} - \delta_{\alpha}{}^{\beta}) \quad (1.22)$$

We also note that, for $m = 1$, the tensors $\overset{\circ}{\Theta}$ and $\mathscr{E}$ have identical mixed components

$$\overset{\circ}{\theta}{}^{\alpha}{}_{\beta} = \hat{\varepsilon}^{\alpha}{}_{\beta} = \frac{1}{2}\,(\delta^{\alpha}{}_{\beta} - \hat{g}^{\alpha\sigma}\overset{\circ}{g}_{\sigma\beta}), \qquad \overset{\circ}{\theta}_{\alpha}{}^{\beta} = \hat{\varepsilon}_{\alpha}{}^{\beta} = \frac{1}{2}\,(\delta_{\alpha}{}^{\beta} - \overset{\circ}{g}_{\alpha\sigma}\hat{g}^{\sigma\beta}) \quad (1.23)$$

From Expression (1.17) it follows that all the tensors corresponding to the tensors $\overset{\circ}{\mathscr{E}}$ and $\mathscr{E}$ can be transformed simultaneously to principal axes.

If the principal values of the finite-strain tensors $\overset{\circ}{\mathscr{E}}$ and $\mathscr{E}$ are denoted by $\overset{\circ}{\epsilon}_i$ and $\hat{\epsilon}_i$ $(i = 1, 2, 3)$, respectively, then the principal values of the tensors of the sequences (1.17) corresponding to the index k are $(1 + 2\overset{\circ}{\epsilon}_i)^k\overset{\circ}{\epsilon}_i$, $(1 - 2\hat{\epsilon}_i)^k\hat{\epsilon}_i$ $(i = 1, 2, 3$; no summation with respect to the index $i)$.

There exists a relation between the principal values $\overset{\circ}{\epsilon}_i$ and $\hat{\epsilon}_i$. Since the mixed components and, consequently, the principal values of the tensors $\overset{\circ}{\mathscr{E}}$ and Θ are identical, we have $\overset{\circ}{\epsilon}_i = \hat{\epsilon}_i(1 - 2\hat{\epsilon}_i)^{-1}$, i.e. the equality which relates the principal values of the finite-strain tensors.

Let us consider now arbitrary tensors of the second order determined by the relations (0.2) and (0.4). These tensors may have identical either the covariant components $\hat{H}_{\alpha\beta}$ and $\overset{\circ}{H}_{\alpha\beta}$ or the contravariant components $\hat{H}^{\alpha\beta}$ and $\overset{\circ}{H}{}^{\alpha\beta}$ or, finally, the mixed components $\hat{H}_{\beta}{}^{\alpha}$ and $\overset{\circ}{H}_{\beta}{}^{\alpha}$ or $\hat{H}_{\alpha}{}^{\beta}$ and $\overset{\circ}{H}_{\alpha}{}^{\beta}$. In the following we shall call the quantities $H^{\alpha}{}_{\beta}$ the mixed components of the first type and the quantities $H_{\beta}{}^{\alpha}$ the mixed components of the second type.

Let us consider the case of the tensors H and $\overset{\circ}{H}$ with equal mixed components of the first type

$$\hat{H}^{\alpha}{}_{\beta} = \mathring{H}^{\alpha}{}_{\beta} = H^{\alpha}{}_{\beta} \tag{1.24}$$

To the tensor H correspond four different tensors in the initial space. The first tensor, with the same mixed components of the first type as the tensor H, is the tensor $\mathring{H}$, according to (1.24). The other tensors, with the same covariant, contravariant or mixed components of the second type as the tensor H, have the mixed components of the first type equal to

$$\mathring{g}^{\alpha\sigma}\hat{g}_{\sigma k}H^{k}{}_{\beta},\ H^{\alpha}{}_{\sigma}\hat{g}^{\sigma k}\mathring{g}_{k\beta},\quad \mathring{g}^{\alpha\sigma}\hat{g}_{\sigma k}H^{k}{}_{e}\hat{g}^{em}\mathring{g}_{m\beta}$$

respectively, and they are thus of the respective forms

$$\mathring{A}\mathring{H},\ \mathring{H}\,\mathring{A}^{-1},\ \mathring{A}\,\mathring{H}\,\mathring{A}^{-1}$$

Similarly, to the tensor $\mathring{H}$ correspond four different tensors H, AH, HA^{-1}, AHA^{-1}, obtained in an analogous way.

Thus, at the first step the tensors corresponding to the tensors H and $\mathring{H}$ may be represented in the form

$$\mathring{A}^{m}\mathring{H}\,\mathring{A}^{n},\qquad A^{m}HA^{n} \tag{1.25}$$

where m and n are the numbers equal to 0, 1 and 0, -1, respectively.

It is easy to verify that, according to the relations (1.4) and (1.24), to an arbitrary tensor of the type $A^{m}HA^{n}$ in the deformed space (m and n are arbitrary integers, positive or negative, or equal to zero) correspond the following tensors in the initial space: the tensor $\mathring{A}^{-m}\mathring{H}\mathring{A}^{-n}$ with the same mixed components of the first type and the tensors $\mathring{A}^{1-m}\mathring{H}\mathring{A}^{-n}$, $\mathring{A}^{-m}\mathring{H}\mathring{A}^{-n-1}$, $\mathring{A}^{1-m}\mathring{H}\mathring{A}^{-n-1}$ with the covariant, contravariant and mixed components of the second type equal to the respective components of the tensor $A^{m}HA^{n}$. Similarly, to an arbitrary tensor $\mathring{A}^{m}H\mathring{A}^{n}$ in the initial space correspond the following tensors in the deformed space: $A^{-m}HA^{-n}$, $A^{1-m}HA^{-n}$, $A^{-m}HA^{-n-1}$, $A^{1-m}HA^{-n-1}$, i.e. the tensors derived are of the same form as the tensors (1.25).

This indicates that the tensors corresponding to the tensor $A^{m}HA^{n}$ are formed in the initial space according to the following rule: the first tensor, with the same mixed components of the first type, is obtained by placing the sign $^{\circ}$ over the factors of the tensor $A^{m}HA^{n}$ and changing the signs of m and n; the other tensors, whose covariant, contravariant or mixed components of the second type are the same as the respective components of the tensor $A^{m}HA^{n}$, are obtained by multiplication of the first tensor, in that order, by the tensor $\mathring{A}$ on the left side, by the tensor $\overset{\circ}{A}^{-1}$ on the right side, and simultaneously by the tensor $\overset{\circ}{A}$ on the left side and the tensor $\overset{\circ}{A}^{-1}$ on the right side. An analogous rule holds for the formation of the tensors in the deformed space which correspond to the tensor of the form $\overset{\circ}{A}^{m}\overset{\circ}{H}\overset{\circ}{A}^{n}$ in the initial space. The application of this rule leads again to the tensors of the form (1.25). Consequently, all the tensors corresponding to the tensors H and $\overset{\circ}{H}$ are of the same form (1.25).

In particular, if H and $\overset{\circ}{H}$ are the metric tensors G and $\overset{\circ}{G}$, then the relations (1.25) result in the relations (1.11) derived previously.

If the tensors H and $\overset{\circ}{H}$ have identical covariant, contravariant, or mixed components of the second type, a similar discussion can show that all the tensors corresponding to H and $\overset{\circ}{H}$ will be also of the form (1.25), and analogous rules for the formation of new tensors can be established. For instance, if the tensors H and $\overset{\circ}{H}$ have identical covariant components $\hat{H}_{\alpha\beta} = \overset{\circ}{H}_{\alpha\beta} = H_{\alpha\beta}$, we have the following rule. To the tensor $A^{m}HA^{n}$ correspond the following four tensors in the initial space: the first tensor, having the same covariant components, is obtained by placing the sign ° over the factors of the tensor $A^{m}HA^{n}$ and changing the signs of m and n; the other tensors, having the same mixed components of the first type, then the second type and, finally, the same contravariant components as the tensor $A^{m}HA^{n}$ are obtained from the first tensor by multiplying on the left side, then the right side and, finally, simultaneously on the left and the right sides by the tensor $\overset{\circ}{A}^{-1}$:

$$\overset{\circ}{A}^{-m}\overset{\circ}{H}\overset{\circ}{A}^{-n},\ \overset{\circ}{A}^{-m-1}\overset{\circ}{H}\overset{\circ}{A}^{-n},\ \overset{\circ}{A}^{-m}\overset{\circ}{H}\overset{\circ}{A}^{-n-1},\ \overset{\circ}{A}^{-m-1}\overset{\circ}{H}\overset{\circ}{A}^{-n-1}.$$

If the tensors H and $\overset{\circ}{H}$ are the finite-strain tensors $\mathscr{E}$ and $\overset{\circ}{\mathscr{E}}$, then, according to the relation (1.15), the products $\overset{\circ}{\mathscr{E}}\overset{\circ}{A}^{\alpha}$ and $\mathscr{E}A^{\alpha}$ (where a is a positive or negative integer) obey the commutative rule, and from the relations (1.25) follow the relations (1.17).

Let us note that the tensors (1.25) are, in general, not symmetric, even if the tensors H and $\overset{\circ}{H}$ are symmetric. Furthermore, it is easy to show that among the tensors obtained at the steps $2k - 1$ and $2k$, $k = 1, 2, \ldots$, of the procedure, only the tensors of the sequence (1.25) for which at least one of the numbers $|m|$ and $|n|$ is equal to k and the other number is an arbitrary natural number from zero to k (including k) are new.

For instance, for $k = 1$, i.e. at the first and the second steps, we have the following new tensors

$$\overset{\circ}{A}\overset{\circ}{H},\quad \overset{\circ}{H}\overset{\circ}{A},\quad \overset{\circ}{A}^{-1}\overset{\circ}{H},\quad \overset{\circ}{H}\overset{\circ}{A}^{-1},\quad \overset{\circ}{A}\overset{\circ}{H}\overset{\circ}{A}^{-1},\quad \overset{\circ}{A}^{-1}\overset{\circ}{H}\overset{\circ}{A},\quad \overset{\circ}{A}\overset{\circ}{H}\overset{\circ}{A},\quad \overset{\circ}{A}^{-1}\overset{\circ}{H}\overset{\circ}{A}^{-1}$$

in the initial space and similar tensors in the deformed space.

2. Let us discuss some applications.

a) In the following, we shall give a generalization [3] of a closed system of the equations of thermo-elasticity of reversible processes, under the assumption that the external heat input and one of the potentials analogous to internal energy, free energy, heat content or thermodynamical potential, related to the unit of mass, are given. In the derivation of the equations of state, the finite strain tensors $\overset{\circ}{\mathscr{E}}$ and $\mathscr{E}$ defined by the relations (1.13) and (1.14) are usually assumed as the characteristics of deformation. We shall show that if the deformation of a medium is described in terms of the tensors $\overset{\circ}{\Theta} = \theta^{\alpha\beta}\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}_\beta$ and $\Theta = \theta^{\alpha\beta}\hat{\partial}_\alpha\hat{\partial}_\beta$, where $\theta^{\alpha\beta}$ are defined by the relations (1.21), an analogous system of the equations of state can be obtained.

The elementary work per unit mass of the internal surface forces is determined by

$$dA^{(i)} = -\frac{1}{\rho} P^{\alpha\beta} d\,\varepsilon_{\alpha\beta} \qquad (\alpha\beta = 1,\ 2,\ 3)$$

where ρ is the density of the medium, $P^{\alpha\beta}$ are contravariant components of the stress tensor $\overset{\circ}{\mathscr{P}} = P^{\alpha\beta}\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}_\beta$ or $\mathscr{P} = P^{\alpha\beta}\hat{\partial}_\alpha\hat{\partial}_\beta$ in the initial and the deformed states, respectively, and $d\epsilon_{\alpha\beta}$ are the increments of the covariant components of the tensors $\overset{\circ}{\mathscr{E}}$ and $\mathscr{E}$ defined by the relations

$$d\varepsilon_{\alpha\beta} = \frac{1}{2}(\nabla_\beta v_\alpha + \nabla_\alpha v_\beta)\,dt$$

$$\nabla_\beta v_\alpha = \frac{\partial v_\alpha}{\partial \xi^\beta} - v_k \hat{\Gamma}_{\alpha\beta}{}^k, \qquad \hat{\Gamma}_{\alpha\beta}{}^k = \frac{1}{2}\hat{g}^{k\sigma}\left(\frac{\partial \hat{g}_{\sigma\alpha}}{\partial \xi^\beta} + \frac{\partial \hat{g}_{\sigma\beta}}{\partial \xi^\alpha} - \frac{\partial \hat{g}_{\alpha\beta}}{\partial \xi^\sigma}\right)$$

with $\bar{v} = v^i\hat{\partial}_i = v_i\hat{\partial}^i$ being the velocity vector of the particles of the medium.

We shall consider that the components of all the tensors discussed and the base vectors are functions of the Lagrangian coordinates ξ^1, ξ^2, ξ^3 and time t.

If the expressions for the time derivatives of the base vectors [2]

$$\frac{d\overset{\circ}{\partial}_i}{dt} = 0, \qquad \frac{d\overset{\circ}{\partial}{}^i}{dt} = 0, \qquad \frac{\hat{d}\partial_i}{dt} = \nabla_i v^\alpha\hat{\partial}_\alpha, \qquad \frac{\hat{d}\partial^i}{dt} = -\nabla_\alpha v^i\hat{\partial}^\alpha \qquad (\alpha,\ i=1,\ 2,\ 3) \tag{2.1}$$

are taken into account, then the relation $d\epsilon_{\alpha\beta} = \hat{g}_{\alpha k}\hat{g}_{e\beta}\,d\theta^{ke}$ can be easily obtained and the work of internal forces expressed in the form

$$dA^{(i)} = -\frac{1}{\rho}\hat{P}_{ke}d\theta^{ke} \qquad (k, e = 1, 2, 3) \tag{2.2}$$

where $\hat{P}_{ke} = P^{\alpha\beta}\hat{g}_{k\alpha}\hat{g}_{\beta e}$ are the covariant components of the tensor $\mathcal{P}$ or the tensor

$$\hat{\mathcal{P}}_3 = \hat{P}_{ke}\hat{\partial}^k\mathring{\partial}^e.$$

We shall consider a medium for which the potentials analogous to the internal energy and the free energy are given by the functions

$$U = U(\mathring{g}_{\alpha\beta}, \ \theta^{\alpha\beta}, S, \mu_1, \ldots, \mu_n, \xi^1, \xi^2, \xi^3)$$
$$F = F(\mathring{g}_{\alpha\beta}, \ \theta^{\alpha\beta}, \ T, \ \mu_1, \ \ldots, \ \mu_n\xi^1, \xi^2, \xi^3) = U - TS$$

Here S is the entropy per unit mass; T is the absolute temperature; μ_i are certain independent physical and chemical parameters characterizing changes of the physical and chemical properties of the medium accompanying mechanical phenomena.

In the case of reversible processes corresponding to all possible variations δU, δS, $\delta\theta^{\alpha\beta}$, $\delta\mu_i$ or δF, δT, $\delta\theta^{\alpha\beta}$, $\delta\mu_i$, the first and the second laws of thermodynamics and Equation (2.2) lead to the relations

$$\begin{aligned} \delta U &= \frac{dU}{dS}\delta S + \frac{\partial U}{\partial\theta^{\alpha\beta}}\delta\theta^{\alpha\beta} + \frac{\partial U}{\partial\mu_i}\delta\mu_i = T\delta S + \frac{1}{\rho}\hat{P}_{\alpha\beta}\delta\theta^{\alpha\beta} \\ \delta F &= \frac{\partial F}{\partial T}\delta T + \frac{\partial F}{\partial\theta^{\alpha\beta}}\delta\theta^{\alpha\beta} + \frac{\partial F}{\partial\mu_i}\delta\mu_i = -S\delta T + \frac{1}{\rho}\hat{P}_{\alpha\beta}\delta\theta^{\alpha\beta} \end{aligned} \tag{2.3}$$

If the admissible increments δS, $\delta\theta^{\alpha\beta}$, $\delta\mu_i$, or δT, $\delta\theta^{\alpha\beta}$, $\delta\mu_i$ are arbitrary, the equations of state, valid for any process, follow:

$$\frac{1}{\rho}\hat{P}_{\alpha\beta} = \left(\frac{\partial U}{\partial\theta^{\alpha\beta}}\right)_{S,\mu_i} = \left(\frac{\partial F}{\partial\theta^{\alpha\beta}}\right)_{T,\mu_i} \tag{2.4}$$

(the remaining relations having the same form as in [3] are omitted). It is clear that the use of the internal energy is more convenient in adiabatic processes, while the use of the free energy is preferable in isothermal processes. Equations (2.4) indicate that the quantities $\tau_{\alpha\beta} = \rho^{-1}\hat{P}_{\alpha\beta}$ may be used instead of $\theta^{\alpha\beta}$ and the systems of variables $\tau_{\alpha\beta}$, S, μ_i, or $\tau_{\alpha\beta}$, T, μ_i may be considered. Then a similar system of equations can be written if the potentials analogous to the heat content and to the thermodynamical potential are taken into account:

$$\Psi = \Psi(\mathring{g}_{\alpha\beta}, \ \tau_{\alpha\beta}, \ S, \mu_1, \ldots, \mu_n, \xi^1, \xi^2, \xi^3) = U - \tau_{\alpha\beta}\theta^{\alpha\beta}$$
$$G = G(\mathring{g}_{\alpha\beta}, \ \tau_{\alpha\beta}, \ T, \ \mu_1, \ldots, \mu_n, \xi^1, \xi^2, \xi^3) = F - \tau_{\alpha\beta}\theta^{\alpha\beta}$$

With the potentials Ψ and G Equations (2.3) assume the form

$$\begin{aligned}\delta\Psi &= \frac{\partial\Psi}{\partial S}\delta S + \frac{\partial\Psi}{\partial\tau_{\alpha\beta}}\delta\tau_{\alpha\beta} + \frac{\partial\Psi}{\partial\mu_i}\delta\mu_i = T\delta S - \theta^{\alpha\beta}\delta\tau_{\alpha\beta}\\ \delta G &= \frac{\partial G}{\partial T}\delta T + \frac{\partial G}{\partial\tau_{\alpha\beta}}\delta\tau_{\alpha\beta} + \frac{\partial G}{\partial\mu_i}\delta\mu_i = -S\delta T - \theta^{\alpha\beta}\delta\tau_{\alpha\beta}\end{aligned} \tag{2.5}$$

Hence, assuming that the variations $\delta S, \delta\tau_{\alpha\beta}, \delta\mu_i$, or $\delta T, \delta\tau_{\alpha\beta}, \delta\mu_i$ are independent, we obtain the equations of state in the form

$$\theta^{\alpha\beta} = -\left(\frac{\partial\Psi}{\partial\tau_{\alpha\beta}}\right)_{S,\mu_i} = -\left(\frac{\partial G}{\partial\tau_{\alpha\beta}}\right)_{T,\mu_i} \tag{2.6}$$

Expressions (2.4) and (2.6) have been derived with the use of the covariant components $\hat{P}_{\alpha\beta}$ of the tensors $\overset{\circ}{P}_3$ and P, and the contravariant components $\theta^{\alpha\beta}$ of the tensors $\overset{\circ}{\Theta}$ and Θ. The equations of state can be also derived using the components of the tensors P and Θ, or the tensors $\overset{\circ}{P}_3$ and $\overset{\circ}{\Theta}$, with different arrangement of indices.

Let us consider the tensors P and Θ. The relations between different components of the tensor Θ are determined by the equalities

$$\theta^{\alpha\beta} = \hat{\theta}_\sigma{}^\alpha \hat{g}^{\sigma\beta} = \hat{\theta}_\sigma{}^\alpha (\overset{\circ}{g}{}^{\sigma\beta} - 2\theta^{\sigma\beta}) \tag{2.7}$$

$$\theta^{\alpha\beta} = \hat{\theta}_{k\sigma}\hat{g}^{\alpha k}\hat{g}^{\sigma\beta} = \hat{\theta}_{k\sigma}(\overset{\circ}{g}{}^{\alpha k} - 2\theta^{\alpha k})(\overset{\circ}{g}{}^{\sigma\beta} - 2\theta^{\sigma\beta}) \tag{2.8}$$

Differentiating these equalities, we obtain the equations

$$d\hat{\theta}_\mu{}^\lambda = (\delta_\alpha{}^\lambda + 2\hat{\theta}_\alpha{}^\lambda)\, d\theta^{\alpha\beta}\hat{g}_{\beta\mu}\,, \quad d\hat{\theta}_{\lambda\mu} = \hat{g}_{\lambda m}\hat{g}_{n\mu}(\delta_\alpha{}^m\delta_\beta{}^n + 2\delta_\alpha{}^m\hat{\theta}_\beta{}^n + 2\hat{\theta}_\beta{}^n\hat{\theta}_\alpha{}^m)\, d\theta^{\alpha\beta} \tag{2.9}$$

We shall consider that the potentials U and F are functions of the mixed or the covariant components of the tensor Θ. Then, using the relations (2.9) and shifting the indices of the components of the tensor P, the equations of state (2.4) can be written in the form

$$\frac{1}{\rho}\hat{P}_\alpha{}^\mu = \frac{1}{\rho}\hat{g}_{\alpha\lambda}P^{\lambda\mu} = (\delta_\alpha{}^\lambda + 2\hat{\theta}_\alpha^\lambda)\left(\frac{\partial U}{\partial\hat{\theta}_\mu{}^\lambda}\right)_{S,\mu_i} = (\delta_\alpha{}^\lambda + 2\hat{\theta}_\alpha{}^\lambda)\left(\frac{\partial F}{\partial\hat{\theta}_\mu{}^\lambda}\right)_{T,\mu_i} \tag{2.10}$$

or

$$\begin{aligned}\frac{1}{\rho}P^{mn} &= \left(\frac{\partial U}{\partial\hat{\theta}_{mn}}\right)_{S,\mu_i} + 2\hat{\theta}_\mu{}^n\left(\frac{\partial U}{\partial\hat{\theta}_{m\mu}}\right)_{S,\mu} + 2\hat{\theta}_\lambda{}^m\left(\frac{\partial U}{\partial\hat{\theta}_{\lambda n}}\right)_{S,\mu_i}\\ &= \left(\frac{\partial F}{\partial\hat{\theta}_{mn}}\right)_{T,\mu_i} + 2\hat{\theta}_\mu{}^n\left(\frac{\partial F}{\partial\hat{\theta}_{m\mu}}\right)_{T,\mu_i} + 2\hat{\theta}_\lambda{}^m\left(\frac{\partial F}{\partial\hat{\theta}_{\lambda n}}\right)_{T,\mu_i}\end{aligned} \tag{2.11}$$

The relations (2.10) and (2.11) give the expressions for the mixed and the contravariant components of the tensor P.

Let us consider now the tensors $\overset{\circ}{P}_3$ and $\overset{\circ}{\Theta}$. The mixed and the contravariant components of the tensor $\overset{\circ}{\Theta}$ are determined by

$$\overset{\circ}{\theta}_\sigma{}^k = \overset{\circ}{g}_{e\sigma}\theta^{ke}, \qquad \overset{\circ}{\theta}_{\sigma\tau} = \overset{\circ}{g}_{\sigma k}\overset{\circ}{g}_{e\tau}\theta^{k\tau} \tag{2.12}$$

Since the components of the metric tensor $\overset{\circ}{G}$ do not depend on time, the differentiation of Expressions (2.12) gives

$$d\overset{\circ}{\theta}_\sigma{}^k = \overset{\circ}{g}_{e\sigma}d\theta^{ke}, \qquad d\overset{\circ}{\theta}_{\sigma\tau} = \overset{\circ}{g}_{\sigma k}\overset{\circ}{g}_{e\tau}d\theta^{ke} \tag{2.13}$$

and, consequently, instead of (2.4) we have the equivalent relations

$$\begin{aligned}
\overset{\circ}{\tau}_k{}^\sigma &= \frac{1}{\rho}\hat{P}_{ke}\overset{\circ}{g}{}^{e\sigma} = \left(\frac{\partial U}{\partial \overset{\circ}{\theta}_\sigma{}^k}\right)_{S,\mu_i} = \left(\frac{\partial F}{\partial \overset{\circ}{\theta}_\sigma{}^k}\right)_{T,\mu_i} \\
\overset{\circ}{\tau}{}^{\sigma\tau} &= \frac{1}{\rho}\hat{P}_{ke}\overset{\circ}{g}{}^{k\sigma}\overset{\circ}{g}{}^{e\tau} = \left(\frac{\partial U}{\partial \overset{\circ}{\theta}_{\sigma\tau}}\right)_{S,\mu_i} = \left(\frac{\partial F}{\partial \overset{\circ}{\theta}_{\sigma\tau}}\right)_{T,\mu_i}
\end{aligned} \tag{2.14}$$

which give the expressions for the mixed and the contravariant components of the tensor $\overset{\circ}{P}_3$. In a similar way, from Equation (2.6) the expressions for the mixed and the contravariant components of the tensor $\overset{\circ}{\Theta}$ follow:

$$\overset{\circ}{\theta}_\sigma{}^k = -\left(\frac{\partial \Psi}{\partial \overset{\circ}{\tau}_k{}^\sigma}\right)_{S,\mu_i} = -\left(\frac{\partial G}{\partial \overset{\circ}{\tau}_k{}^\sigma}\right)_{T,\mu_i}, \quad \overset{\circ}{\theta}_{\sigma\tau} = -\left(\frac{\partial \Psi}{\partial \overset{\circ}{\tau}{}^{\sigma\tau}}\right)_{S,\mu_i} = -\left(\frac{\partial G}{\partial \overset{\circ}{\tau}{}^{\sigma\tau}}\right)_{T,\mu_i} \tag{2.15}$$

It is easy to verify that the relations (2.10) and the first of the relations (2.14) can be written in the form

$$\begin{aligned}
\frac{1}{\rho}\overset{\circ}{P}_m{}^\sigma &= \left(\frac{\partial U}{\partial \overset{\circ}{\varepsilon}_\sigma{}^m}\right)_{S,\mu_i} = \left(\frac{\partial F}{\partial \overset{\circ}{\varepsilon}_\sigma{}^m}\right)_{T,\mu_i} \\
\frac{1}{\rho}\hat{P}_k{}^e &= (\delta_\sigma{}^e - 2\hat{\varepsilon}_\sigma{}^e)\left(\frac{\partial U}{\partial \hat{\varepsilon}_\sigma{}^k}\right)_{S,\mu_i} = (\delta_\sigma{}^e - 2\hat{\varepsilon}_\sigma{}^e)\left(\frac{\partial F}{\partial \hat{\varepsilon}_\sigma{}^k}\right)_{T,\mu_i}
\end{aligned} \tag{2.16}$$

where $\overset{\circ}{P}_m{}^\sigma = \overset{\circ}{g}_{mk}P^{k\sigma}$, $\overset{\circ}{\epsilon}_\sigma{}^m = \overset{\circ}{g}{}^{mk}\epsilon_{k\sigma}$, $\hat{\epsilon}_\sigma{}^k = \hat{g}^{k\tau}\epsilon_{\tau\sigma}$ are mixed components of the tensors $\overset{\circ}{P}$, $\overset{\circ}{\mathscr{E}}$, $\hat{\mathscr{E}}$, respectively. The relations (2.16) were derived in [3].

b) In the theory of finite deformation it is more convenient to consider instead of the tensor $P = P^{\alpha\beta}\hat{\partial}_\alpha\hat{\partial}_\beta$ the tensor $\sigma = \sigma^{\alpha\beta}\hat{\partial}_\alpha\hat{\partial}_\beta$, $\sigma^{\alpha\beta} = 1/\rho P^{\alpha\beta}$, because the quantities $\sigma^{\alpha\beta}$ have a potential.

We shall introduce four different definitions of the rates of the tensor σ, and we shall show that one of them coincides with the definition proposed by Truesdell [4]. In addition to the convected coordinate system, we introduce a fixed coordinate system in the deformed space x^1, x^2, x^3, with the base vectors ∂_i, ∂^i, $i = 1, 2, 3$. Without limiting the generality of this discussion, we assume that this new reference system is Cartesian. To the tensor

$$\sigma = \sigma^{\alpha\beta}\hat{\partial}_\alpha\hat{\partial}_\beta = \hat{\sigma}_\beta{}^\alpha\hat{\partial}_\alpha\hat{\partial}^\beta = \hat{\sigma}_\alpha{}^\beta\hat{\partial}^\alpha\hat{\partial}_\beta = \hat{\sigma}_{\alpha\beta}\hat{\partial}^\alpha\hat{\partial}^\beta = \sigma'^{\alpha\beta}\partial_\alpha\partial_\beta \tag{2.17}$$

correspond four different tensors in the initial space

$$\overset{\circ}{\sigma}_1 = \sigma^{\alpha\beta}\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}_\beta, \quad \overset{\circ}{\sigma}_2 = \hat{\sigma}^\alpha{}_\beta\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}^\beta, \quad \overset{\circ}{\sigma}_3 = \hat{\sigma}_\alpha{}^\beta\overset{\circ}{\partial}^\alpha\overset{\circ}{\partial}_\beta, \quad \overset{\circ}{\sigma}_4 = \hat{\sigma}_{\alpha\beta}\overset{\circ}{\partial}^\alpha\overset{\circ}{\partial}^\beta$$

The different rates $\overset{\circ}{V}_i$, $i = 1, 2, 3, 4$ of the tensor σ are defined as the time derivatives of the tensors $\overset{\circ}{\sigma}_i$:

$$\overset{\circ}{V}_1 = \frac{d\sigma^{\alpha\beta}}{dt}\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}_\beta, \qquad \overset{\circ}{V}_2 = \frac{d\hat{\sigma}^\alpha{}_\beta}{dt}\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}^\beta, \qquad \overset{\circ}{V}_3 = \frac{d\hat{\sigma}_\alpha{}^\beta}{dt}\overset{\circ}{\partial}^\alpha\overset{\circ}{\partial}_\beta, \qquad \overset{\circ}{V}_4 = \frac{d\hat{\sigma}_{\alpha\beta}}{dt}\overset{\circ}{\partial}^\alpha\overset{\circ}{\partial}^\beta$$

The expressions for the components of these tensors can be obtained by differentiating the tensor σ in the invariant forms (2.17). Using Expressions (2.1) for the time derivatives of the base vectors $\hat{\partial}_i$ and $\hat{\partial}^i$ and taking into account the fact that the reference system is Cartesian and consequently

$$\frac{d\partial^i}{dt} = \frac{d\partial^i}{dt} = 0$$

we obtain from the differentiation of (2.17)

$$\begin{aligned}\frac{d\sigma}{dt} &= \left(\frac{d\sigma^{\alpha\beta}}{dt} + \sigma^{k\beta}\nabla_k v^\alpha + \sigma^{\alpha k}\nabla_k v^\beta\right)\hat{\partial}_\alpha\hat{\partial}_\beta = \left(\frac{d\hat{\sigma}^\alpha{}_\beta}{dt} + \hat{\sigma}^k{}_\beta\nabla_k v^\alpha - \hat{\sigma}^\alpha{}_k\nabla_\beta v^k\right)\hat{\partial}_\alpha\hat{\partial}^\beta \\ &= \left(\frac{d\hat{\sigma}_\alpha{}^\beta}{dt} - \hat{\sigma}_k^\beta\nabla_\alpha v^k + \hat{\sigma}_\alpha{}^k\nabla_k v^\beta\right)\hat{\partial}^\alpha\hat{\partial}_\beta = \left(\frac{d\hat{\sigma}_{\alpha\beta}}{dt} - \hat{\sigma}_{k\beta}\nabla_\alpha v^k - \hat{\sigma}_{\alpha k}\nabla_\beta v^k\right)\hat{\partial}^\alpha\hat{\partial}^\beta \\ &= \frac{1}{\rho}\left(\frac{dP'^{\alpha\beta}}{dt} + P'^{\alpha\beta}\nabla_k v'^k\right)\partial_\alpha\partial_\beta\end{aligned}$$

Here the equality $\sigma'^{\alpha\beta} = 1/\rho\, P'^{\alpha\beta}$ and the continuity equation

$$\frac{d\rho}{dt} + \rho\nabla_k v'^k = 0, \quad \bar{v} = v^k\hat{\partial}_k = v'^k\partial_k$$

have been utilized.

Assuming that at the instant of time being considered the base vectors $\hat{\partial}_i$ and ∂_i coincide, we obtain the relations which hold in the curvilinear coordinate systems:

$$\begin{aligned}\rho\frac{d\sigma^{\alpha\beta}}{dt} &= \frac{dp'^{\alpha\beta}}{dt} + P^{\alpha\beta}\nabla_k v^k - P^{k\beta}\frac{\partial v^\alpha}{\partial x^k} - P^{\alpha k}\frac{\partial v^\beta}{\partial x^k} \\ \rho\frac{d\hat{\sigma}^\alpha{}_\beta}{dt} &= \frac{dP'^\alpha{}_\beta}{dt} + P^\alpha{}_\beta\nabla_k v^k - P^\alpha{}_k\frac{\partial v^\alpha}{\partial x^k} + P_k{}^\alpha\frac{\partial v^k}{\partial x^\beta}\end{aligned} \tag{2.18}$$

$$\rho \frac{d\hat{\sigma}_{\alpha}{}^{\beta}}{dt} = \frac{dP'_{\alpha}{}^{\beta}}{dt} + P_{\alpha}{}^{\beta} \nabla_k v^k + P^k{}_{\beta} \frac{\partial v^k}{\partial x^\alpha} - P_{\alpha}{}^{k} \frac{\partial v^\beta}{\partial x^k}$$

$$\rho \frac{d\hat{\sigma}_{\alpha\beta}}{dt} = \frac{dP'_{\alpha\beta}}{dt} + P_{\alpha\beta} \nabla_k v^k + P_{k\beta} \frac{\partial v^k}{\partial x^\alpha} + P_{\alpha k} \frac{\partial v^k}{\partial x^\beta}$$

In a Cartesian coordinate system the positions of the indices are immaterial and the first of Expressions (2.18) coincides with the definition of the rate of stress proposed by Truesdell.

It is easy to see that the invariants of the tensor σ coincide with the invariants of the tensors $\overset{\circ}{\sigma}_2$ and $\overset{\circ}{\sigma}_3$, and they are different from the invariants of the tensors $\overset{\circ}{\sigma}_1$ and $\overset{\circ}{\sigma}_4$. Obviously, the time-derivatives of the invariants of the tensors σ, $\overset{\circ}{\sigma}_2$, $\overset{\circ}{\sigma}_3$ become equal to zero with the tensors $\overset{\circ}{V}_2$ or $\overset{\circ}{V}_3$, and the time-derivatives of the invariants of the tensors $\overset{\circ}{\sigma}_1$ or $\overset{\circ}{\sigma}_4$ become equal to zero with the tensors $\overset{\circ}{V}_1$ or $\overset{\circ}{V}_4$, respectively,

We note that with the relations

$$\overset{\circ}{\sigma}_2 = \overset{\circ}{\sigma}_1 \overset{\circ}{A}, \qquad \overset{\circ}{\sigma}_3 = \overset{\circ}{A} \overset{\circ}{\sigma}_1, \qquad \overset{\circ}{\sigma}_4 = \overset{\circ}{A} \overset{\circ}{\sigma}_1 \overset{\circ}{A}$$

the following relations for the rates of the tensor σ can be easily derived:

$$\overset{\circ}{V}_2 = \overset{\circ}{V}_1 (\overset{\circ}{G} + 2\overset{\circ}{E}) + 2\overset{\circ}{\sigma}_1 \overset{\circ}{e}, \qquad \overset{\circ}{V}_3 = 2\overset{\circ}{e}\overset{\circ}{\sigma}_1 + (\overset{\circ}{G} + 2\overset{\circ}{E}) \overset{\circ}{V}_1$$

$$\overset{\circ}{V}_4 = 2\overset{\circ}{e}\overset{\circ}{\sigma}_1 (\overset{\circ}{G} + 2\overset{\circ}{E}) + (\overset{\circ}{G} + 2\overset{\circ}{\varepsilon}) \overset{\circ}{V}_1 (\overset{\circ}{G} + 2\overset{\circ}{E}) + 2 (\overset{\circ}{G} + 2\overset{\circ}{E}) \overset{\circ}{\sigma}_1 \overset{\circ}{e}$$

where $\overset{\circ}{e}$ is the deformation-rate tensor.

The problems considered show that the proposed tensorial characteristics may be used in the discussion of different questions of the theory of finite deformations.

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